From Game Theory to Graph Theory: A Bilevel Journey

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OR 2018, Brussels
EURO Plenary
Stackelberg Games

- **Two-player sequential-play game**: LEADER and FOLLOWER

- LEADER moves before FOLLOWER - first mover advantage

- **Perfect information**: both agents have perfect knowledge of each other's strategy

- **Rationality**: agents act optimally, according to their respective goals

- LEADER takes FOLLOWERS’s optimal response into account

- **Optimistic vs Pessimistic**: when FOLLOWER has multiple optimal responses
Stackelberg Games

- **Two-player sequential-play game**: LEADER and FOLLOWER
- LEADER moves first, Follower moves second
  - **First mover advantage**
- Perfect information: both agents have perfect knowledge of each others strategy
- Rationality: agents act optimally, according to their respective goals
  - In any given situation a decision maker always chooses the action which is the best according to his/her preferences (a.k.a. rational play).
- LEADER takes FOLLOWER’s optimal response into account
- **Optimistic vs Pessimistic**: when FOLLOWER has multiple optimal responses
Stackelberg Games

• Introduced in economy by v. Stackelberg in 1934

• 40 years later introduced in Mathematical Optimization → Bilevel Optimization

A Convex Programming Model for Optimizing SLBM Attack of Bomber Bases

Jerome Bracken and James T. McGill
Institute for Defense Analyses, Arlington, Virginia
(Received July 30, 1970)

This paper formulates a convex programming model allocating submarine-launched ballistic missiles (SLBMs) to launch areas and providing simultaneously an optimal targeting pattern against a specified set of bomber bases. Flight times of missiles from launch areas to bases vary and targets decrease in value over time. A nonseparable concave objective function is given for expected destruction of bombers. An example is presented.
Applications: Pricing

Two competitive agents act in a hierarchical way with different/conflicting objectives

- **Pricing**: operator sets tariffs, and then customers choose the cheapest alternative

- Tariff-setting, toll optimization (Labbé et al., 1998; Brotcorne et al., 2001)
- Network Design and Pricing (Brotcorne et al., 2008)
- Survey (van Hoesel, 2008)
Applications: Interdiction

Canada and the Transcontinental Drug Links

Canada and the Transcontinental Drug Links

Canadian police conducted several simultaneous raids on suspected drug traffickers in Newfoundland and Quebec provinces Oct. 11, arresting two dozen people and seizing marijuana, cocaine, weapons, cash and property. The drug-trafficking ring, which Canadian authorities believe was operated by the Quebec-based Hell’s Angels motorcycle/crime gang, could have smuggled the cocaine into Canada from South America via Mexico and the United States.

More than 70 members of the Royal Newfoundland Constabulary and Quebec’s Provincial Biker Enforcement Unit carried out the raids, which represented the culmination of an 18-monthlong investigation dubbed Operation Roadrunner. The arrests were made near St. John’s in Newfoundland and near the towns of Laval and La Tuque in Quebec. In Newfoundland, authorities seized $300,000 in cash, 51 pounds of marijuana and 19 pounds of cocaine, as well as vehicles, weapons and computers. In Quebec, $170,000 and four houses were seized.

source: banderasnews.com
Applications: Interdiction

Canada and the Transcontinental Drug Links

Strategic Forecasting Inc
go to original

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The jungles of South America, where cocaine is produced, seem a long way from the St. Lawrence River. Using a sophisticated shipment and distribution network, however, criminal and militant organizations can cover the distance in a few days.

• Monitoring / halting an adversary’s activity on a network
  • Maximum-Flow Interdiction
  • Shortest-Path Interdiction

• Action:
  • Destruction of certain nodes / edges
  • Reduction of capacity / increase of cost on certain edges

• The problems are NP-hard! Survey (Collado and Papp, 2012)

• Uncertainties:
  • Network characteristics
  • Follower’s response

source: banderasnews.com
Bilevel Optimization

General bilevel optimization problem

\[
\min_{x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}} F(x, y) \\
G(x, y) \leq 0
\]

Both players may involve integer decision variables, functions can be non-linear, non-convex...
Bilevel Optimization

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\begin{align*}
\min_{x \in \mathbb{R}^{m_1}, y \in \mathbb{R}^{n_2}} & \quad F(x, y) \\
G(x, y) & \leq 0
\end{align*}
\]

\[
y \in \arg \min_{y' \in \mathbb{R}^{n_2}} \{ f(x, y') : g(x, y') \leq 0 \}
\]

Stephan Dempe
Bilevel optimization: theory, algorithms and applications

Both players may involve integer decision variables, functions can be non-linear, non-convex…

References: 1362
Hierarchy of bilevel optimization problems

Bilevel Optimization

General Case

\[ g(x, y') \leq 0 \]

Interdiction-Like

\[ y'_j \leq 1 - x_j \]

Under Uncertainty, Multiobjective, inf dim spaces, ...

Follower: Convex

Jeroslow, MP, 1985

NP-hard (LP+LP)

Follower: Non-Convex

Follower: (M)ILP

Network Interdiction (LP)

Follower: Convex

Follower: Non-Convex

Follower: (M)ILP

... This talk!

Fischetti, Ljubic, Monaci, Sinnl, OR, 2017: Branch&Cut
About our journey

• With **sparse MILP formulations**, we can now solve to optimality:
  • Covering Facility Location (Cordeau, Furini, Ljubic, 2018): **20M clients**
    • **Code**: https://github.com/fabiofurini/LocationCovering
  • Competitive Facility Location (Ljubic, Moreno, 2017): **80K clients (nonlinear)**
  • Facility Location Problems (Fischetti, Ljubic, Sinnl, 2016): 2K x 10K instances
  • Steiner Trees (DIMACS Challenge, 2014): 150k nodes, 600k edges

• **Common to all**: **Branch-and-Benders-Cut**

Is there a way to exploit sparse formulations along with Branch-and-Cut for bilevel optimization?
Problems addressed today...

• **Interdiction-Like Problems:** LEADER ”interdicts” FOLLOWER by removing some “objects”. Both agents play pure strategies.

• FOLLOWER solves a combinatorial optimization problem (mostly, an NP-hard problem!). One could build a payoff matrix (exponential in size!).

• We propose a generic **Branch-and-Interdiction-Cuts framework** to efficiently solve these problems in practice!
  • Assuming **monotonicity property** for FOLLOWER: **interdiction cuts** (facet-defining)
  • Computationally outperforming state-of-the-art

• Draw a connection to some problems in Graph Theory
Based on a joint work with...


• M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl: Interdiction Games and Monotonicity, with Application to Knapsack Problems, *INFORMS Journal on Computing*, in print, 2018

• F. Furini, I. Ljubic, P. San Segundo, S. Martin: The Maximum Clique Interdiction Game, *Optimization Online*, 2018

• F. Furini, I. Ljubic, E. Malaguti, P. Paronuzzi: On Integer and Bilevel Formulations for the k-Vertex Cut Problem, submitted, 2018
Branch-and-Interdiction-Cut

A gentle introduction
Interdicting Communities in a Network

Critical Nodes: disconnect the network „the most“
Survey: Lalou et al. (2018)

Defender-Attacker Game
LEADER: eliminates the nodes
FOLLOWER: builds communities

Research question: assuming that one can prevent $k$ members of doing criminal activities, what is the size of the largest community that will remain?

After studying the lives of 172 terrorists, Sageman found the most common factors driving them are the social ties. Communities in social networks are often characterized as densely connected subgraphs.
Hamburg Cell: Max-Clique Interdiction

$k=0$

$k=4$
Hamburg Cell: Max-Clique Interdiction

k=0

k=8
Bilevel Integer Program

\[
\begin{align*}
\min & \quad d^T y \\
\text{subject to} & \quad b^T x \leq B_D \\
y & \in \arg\max\{d^T y : \quad y_i \leq 1 - x_i, \quad i \in N \\
& \quad y \in Y\} \\
x_i & \text{ binary, } \quad i \in N
\end{align*}
\]

\[
\begin{align*}
\min & \quad w \\
\text{subject to} & \quad b^T x \leq B_D \\
w & \geq \max\{d^T y : \quad y_i \leq 1 - x_i, \quad i \in I \\
& \quad y \in Y\} \\
x_i & \text{ binary, } \quad i \in I
\end{align*}
\]

Value Function

\[\Phi(x)\]
**Value Function Reformulation**

\[
\min_{x \in \mathbb{R}^{|N|}, w \in \mathbb{R}} w \\
\text{subject to} \\
w \geq \Phi(x) \\
b^T x \leq B_D \\
x_i \text{ binary, } i \in N
\]

**INTERDICTION: Min-max**

\[
\min_{x \in \mathbb{R}^{|N|}} b^T x \\
\text{subject to} \\
K \geq \Phi(x) \\
x_i \text{ binary, } i \in N
\]

**BLOCKING: Min-num or Min-sum**
Value Function Reformulation

\[
\min_{x \in \mathbb{R}^{|N|}, w \in \mathbb{R}} w \\
\quad w \geq \Phi(x) \\
b^T x \leq B_D \\
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**INTERDICTION:** Min-max

**BLOCKING:** Min-num or Min-sum

**GENERAL IDEA:**

- Benders-Like Reformulation: \( y \) variables are projected out!
- If function \( \Phi(x) \) could be “convexified” (using linear functions in \( x \)), we would obtain an MILP!
- To be solved in a branch-and-cut fashion
How to convexify the value function?
**Convexification**

**Observation:** Given $x$, for the optimal follower’s response it holds:

$$x_j + y_j \leq 1 \implies x_j y_j = 0 \quad j \in N$$

Instead of solving:

$$\Phi(x) = \max_{y \in \mathbb{R}^{n_2}} \, d^T y$$

$$0 \leq y_j \leq 1 - x_j, \quad \forall j \in N$$

$$y \in Y$$

$$Y = \{ y \in \mathbb{R}^{n_2} : \quad Qy \leq q_0, \quad y_j \text{ integer } \forall j \in J_y \}.$$ 

Wood (2011) proposes to move $x$ into the objective function and find the penalties $M_j$, such that we can equivalently solve:

$$\Phi(x) = \max_{y \in \mathbb{R}^{n_2}} \{ d^T y - \sum_{j \in N} M_j x_j y_j \} = \max_{\hat{y} \in \text{conv}(Y)} \{ d^T \hat{y} - \sum_{j \in N} M_j x_j \hat{y}_j \}$$
Convexification $\rightarrow$ Benders-Like Reformulation

Benders-Like Reformulation

\[
\begin{align*}
\min_{x \in \mathbb{R}^n, w \in \mathbb{R}} & \quad w \\
\text{s.t.} & \quad w \geq d^T \hat{y} - \sum_{j \in N} M_j x_j \hat{y}_j, \quad \forall \hat{y} \in \hat{Y} \\
& \quad Ax \leq b \\
& \quad x_j \text{ integer,} \quad \forall j \in J_x \\
& \quad x_j \text{ binary,} \quad \forall j \in N.
\end{align*}
\]

The choice of $M_j$ is crucial:

- If FOLLOWER solves an LP: Wood (2011), $M_j$ is the upper bound of the dual variable.
- If FOLLOWER solves the KNAPSACK PROBLEM: Caprara et al. (2016), De Negre (2011), $M_j = d_j$.
- In general: OPEN QUESTION.
If the follower satisfies monotonicity property...

\[ Y = \{ y \in \mathbb{R}^{n_2} \mid Qy \leq q_0, \quad y_j \text{ integer } \forall j \in J_y \}. \]

**Theorem (Fischetti, Ljubić, Monaci, Sinnl, 2018)**

For Interdiction Games with Monotonicity, \( M_j = d_j \), i.e., we have:

\[
\min_{x \in \mathbb{R}^{n_1}, w \in \mathbb{R}} \quad w \\
\text{s.t.} \quad w \geq \sum_{j \in N} d_j \hat{y}_j (1 - x_j) \quad \forall \hat{y} \in \hat{Y} \\
Ax \leq b \\
x_j \text{ integer, } \forall j \in J_x \\
x_j \text{ binary, } \forall j \in N.
\]

**Downward Monotonicity:** \( Q \geq 0 \)

If \( \hat{y} \) is a feasible follower and \( y' \) satisfies integrality constraints and \( 0 \leq y' \leq \hat{y} \), then \( y' \) is also feasible.
If the follower satisfies monotonicity property...

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- max-knapsack (set packing)
- max-clique
- max-relaxed-clique (\( s \)-plex: degree, \( s \)-clique: distance, \( s \)-bundle: connectivity)
A Careful Branch-and-Interdiction-Cut Design

- **Separation:** finding the best FOLLOWER’s response for a given $x^*$. **NP-hard, in general.**

- A good **balance** between “lazy cut separation” (integer points only) and “user cut separation” (fractional points).
A Careful Branch-and-Interdiction-Cut Design

- **Separation:** finding the best FOLLOWER’s response for a given $x^*$. **NP-hard, in general.**
- A good **balance** between “lazy cut separation” (integer points only) and “user cut separation” (fractional points).
- **Crucial:** specialized procedures/algorithms for FOLLOWER’s sub-problem (if available).
- **Combinatorial** algorithms for LOWER and UPPER BOUNDS.
- Efficient **PREPROCESSING** techniques.

→ Branch-and-Interdiction-Cut
A Careful Branch-and-Interdiction-Cut Design

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- **Combinatorial** algorithms for LOWER and UPPER BOUNDS.

- Efficient **PREPROCESSING** techniques.

- Under **monotonicity property**: Interdiction cuts are facet-defining or could be lifted, otherwise.

- Resulting in general in **strong LP-relaxation bounds**.
Max-Clique-Interdiction on Large-Scale Networks

| Graph      | $|V|$     | $|E|$           | $\omega$ [s] | $k = 0.005 \cdot |V|$ | $|V_p|$ | $k = 0.01 \cdot |V|$ | $|V_p|$ | Eliminated by preprocessing |
|------------|----------|----------------|-------------|---------------------|--------|---------------------|--------|--------------------------------|
| socfb-Ullinois | 30,795   | 1,264,421      | 0.5         | 24.4                | 10,456 | 41.6                | 8290   | |
| ia-email-EU   | 32,430   | 54,397         | 0.0         | 0.6                 | 30,375 | 0.5                 | 29,212 | |
| ia-enron-large | 33,696   | 180,811        | 0.0         | 2.2                 | 27,791 | 29.5                | 26,651 | |
| socfb-UF     | 35,111   | 1,465,654      | 0.3         | 17.8                | 14,264 | 87.8                | 10,708 | |
| socfb-Texas84 | 36,364   | 1,590,651      | 0.3         | 24.6                | 10,706 | 74.3                | 8,704  | |
| fe-tooth     | 78,136   | 452,591        | 0.5         | 18.9                | 7      | 19.0                | 7      | |
| sc-pkustk11  | 87,804   | 2,565,054      | 1.1         | 70.7                | 2,712  | 57.1                | 2,712  | |
| ia-wiki-Talk | 92,117   | 360,767        | 0.2         | 49.2                | 72,678 | 87.4                | 72,678 | |
| sc-pkustk13  | 94,893   | 3,260,967      | 1.3         | 724.9               | 2,360  | 879.2               | 2,354  | |

Furini, Ljubic, Martin, San Segundo (2018)
## Max-Clique-Interdiction on Large-Scale Networks

<table>
<thead>
<tr>
<th>#variables</th>
<th>Max-Clique Solver</th>
<th>Furini, Ljubic, Martin, San Segundo (2018)</th>
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<td>San Segundo et al. (2016)</td>
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B&IC Ingredients

% of instances

\( \tau \)

CLIQUE-INTER
CLIQUE-INTER (no bounds)
CLIQUE-INTER (no lifting)
Basic CLIQUE-INTER with IMCQ
Basic CLIQUE-INTER with CPLEX
Comparison with the state-of-the-art MILP bilevel solver

| $|V|$ | # | # solved | time | exit gap | root gap | # solved | time  | exit gap | root gap |
|-----|---|---------|------|----------|----------|---------|-------|----------|----------|
| 50  | 44| 44      | 0.01 | -        | 0.16     | 28      | 68.58 | 6.44     | 8.50     |
| 75  | 44| 44      | 1.45 | -        | 0.41     | 14      | 120.19| 9.47     | 10.91    |
| 100 | 44| 37      | 9.30 | 1.00     | 0.98     | 7       | 164.42| 12.65    | 13.11    |
| 125 | 44| 35      | 13.43| 1.33     | 1.20     | 2       | 135.33| 13.88    | 14.73    |
| 150 | 44| 33      | 27.23| 1.91     | 1.43     | 1       | 397.52| 16.42    | 16.39    |
B&IC works well even if follower has more decision variables, as long as monotonicity holds for interdicted variables.
The result can be further generalized

Relevant Operations Research applications. Two companies competing at the market for customers.

- **LEADER**: established on the market,
- **FOLLOWER**: a newcomer who wants to disrupt the market.

LEADER wants to keep the customers by providing them coupons, vouchers. FOLLOWER is solving a profit-maximization problem:

- **NETWORK DESIGN**: prize-collecting Steiner tree
- **LOGISTICS**: orienteering problems
- **FACILITY LOCATION**: profit maximization variant

Fischetti, Ljubic, Monaci, Sinnl (2018)
And what about Graph Theory?
• **Property:** A set of vertices is a *vertex cover* if and only if its complement is an independent set

• **Vertex Cover as a Blocking Problem:**
  • **LEADER:** interdicts (removes) the nodes.
  • **FOLLOWER:** maximizes the size of the largest connected component in the remaining graph.
  • Find the smallest set of nodes to interdict, so that FOLLOWER’s optimal response is at most one.
The $k$-Vertex-Cut Problem

- A set of vertices is a vertex $k$-cut if upon its removal the graph contains at least $k$ components.
- The $k$ Vertex-Cut Problem: Find a vertex $k$-cut of minimum cardinality/weight.

Open question: Is there an ILP formulation in the natural space of variables?
The k-Vertex-Cut Problem

- A set of vertices is a vertex $k$-cut if upon its removal the graph contains at least $k$ components.

- **The $k$ Vertex-Cut Problem:** Find a vertex $k$-cut of minimum cardinality/weight.

- Influential nodes in a diffusion model for social networks, Kempe et al. (2005)

- Decomposition method for linear equation systems, e.g. GCG solver (Bastubbe, Lübbecke, 2017)

**Open question:** Is there an ILP formulation in the natural space of variables?
**K-Vertex-Cut**

**Property:** A graph $G$ has at least $k$ (not empty) components if and only if any cycle-free subgraph of $G$ contains at most $|V| - k$ edges.

Example with $|V| = 9$ and $k = 3$:
**Property:** A graph $G$ has at least $k$ (not empty) components if and only if any cycle-free subgraph of $G$ contains at most $|V| - k$ edges.

Example with $|V| = 9$ and $k = 3$:

Stackelberg game:

- **LEADER:** searches the smallest subset of nodes to delete;
- **FOLLOWER** maximizes the size of the cycle-free subgraph on the remaining graph.
**k-Vertex-Cut: Benders-like reformulation**

\[
\min \sum_{v \in V} x_v \\
\Phi(x) \leq |V| - \sum_{v \in V} x_v - k \\
x_v \in \{0, 1\} \quad v \in V.
\]

The value function reformulation

The following **Natural Space Formulation**, is a valid model for the \(k\)-vertex cut problem (Furini et al. 2018):

\[
\min \sum_{v \in V} x_v \\
\sum_{v \in V} [\deg_T(v) - 1] x_v \geq k - |V| + |E(T)| \\
x_v \in \{0, 1\} \quad v \in V.
\]

\(T \in T,\)
k-Vertex-Cut: Benders-like reformulation

Furini et al. (2018)
Prev. STATE-OF-THE-ART

Compact model

Branch-and-Interdiction-Cut
Conclusions.

And some directions for the future research.
Takeaways

• Bilevel optimization: very difficult!

• **Branch-and-Interdiction-Cuts** can work very well in practice:
  • Problem reformulation in the **natural space of variables** („thinning out“ the heavy MILP models)
  • **Tight „interdiction cuts“** (monotonicity property)
  • **Crucial:** Problem-dependent (combinatorial) separation strategies, preprocessing, combinatorial poly-time bounds

• Many **graph theory problems** (node-deletion, edge-deletion) could be solved efficiently, when **approached from the bilevel-perspective**
Possible directions for future research

• **Bilevel Optimization**: a better way of *integrating customer behaviour* into decision making models

• Generalizations of **Branch-and-Interdiction-Cuts** for:
  • *Non-linear* follower functions
  • *Submodular* follower functions
  • Interdiction problems *under uncertainty*
  • ...

• Extensions to **Defender-Attacker-Defender (DAD) Models** (*trilevel games*)

• Benders-like decomposition for general mixed-integer bilevel optimization
Thank you for your attention!

References:


  SOLVER: [https://msinnl.github.io/pages/bilevel.html](https://msinnl.github.io/pages/bilevel.html)

• M. Fischetti, I. Ljubic, M. Monaci, M. Sinnl: Interdiction Games and Monotonicity, with Application to Knapsack Problems, *INFORMS Journal on Computing*, in print, 2018

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Literature


• J.F. Cordeau, F. Furini, I. Ljubic. Benders Decomposition for Very Large Scale Partial Set Covering and Maximal Covering Problems, submitted, 2018


• M. Fischetti, I. Ljubic, M. Sinnl: Redesigning Benders Decomposition for Large Scale Facility Location, Management Science 63(7): 2146-2162, 2017
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• I. Ljubic, E. Moreno: Outer approximation and submodular cuts for maximum capture facility location problems with random utilities, European Journal of Operational Research 266(1): 46-56, 2018


